HW #5 KEY

$$P_i = \frac{|\langle a_i|\psi\rangle|^2}{\langle\psi|\psi\rangle}$$

$|\langle a_i|\psi\rangle|^2$ → probability of ψ collapsing onto the i th state upon measurement

$\langle\psi|\psi\rangle$ → 1 since ψ is normalized

Find what is <u>N</u>

$\langle\psi|\psi\rangle = 1$

$$\Rightarrow \int_0^L N^2 x^2 (x-L)^2 dx = 1$$

$$N^2 \int_0^L x^2(x-L)^2 dx = 1$$

$$N^2 \int_0^L x^2(x^2 - 2Lx + L^2) dx = 1$$

$$N^2\left[\frac{x^5}{5} - \frac{2Lx^{\cancel{3}4}}{4} + \frac{L^2x^3}{3}\right]_0^L = 1$$

$$N^2\left[\frac{L^5}{5} - \frac{2L^5}{4} + \frac{L^5}{3}\right] = 1$$

$$N^2\left[\frac{12L^5 - 30L^5 + 20L^5}{60}\right] = 1$$

$$N^2 = \left(\frac{60}{2L^5}\right)$$

$$N = \sqrt{\frac{30}{L^5}}$$

$$\psi(x) = \sqrt{\frac{30}{L^5}}\, x(x-L)$$

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \Leftarrow \text{PIB eigenfunctions}$$

$$c_n = \langle \phi_n | \psi \rangle = \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \sqrt{\frac{30}{L^5}}\, x(x-L)\, dx$$

$$= \sqrt{\frac{60}{L^6}} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \left[x^2 - Lx\right] dx$$

$$= \frac{2\sqrt{15}}{L^3} \left[\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right) dx - L \int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx \right]$$

$$= \frac{2\sqrt{15}}{L^3} \Bigg\{ \left[2\left(\frac{L}{n\pi}\right)^2 x \sin\left(\frac{n\pi}{L} x\right) - \left\{ \frac{(n\pi x/L)^2 - 2}{(n\pi/L)^3} \cos\left(\frac{n\pi x}{L}\right) \right. \right]_0^L$$

$$- L \left[\left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right) - \frac{Lx}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \right]_0^L \Bigg\}$$

$$= \frac{2\sqrt{15}}{L^3} \left\{ -\frac{((n\pi)^2 - 2)L^3}{(n\pi)^3} \cos(n\pi) - \frac{2L^3}{(n\pi)^3} + \frac{L^3}{n\pi} \cos n\pi \right\}$$

$$= \frac{4\sqrt{15}}{(n\pi)^3} \left(-1 + \cos(n\pi)\right)$$

$$\Rightarrow c_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{8\sqrt{15}}{(n\pi)^3} & \text{if } n \text{ is odd} \end{cases}$$

$$\Rightarrow P_n = \frac{|\langle \phi_n | \psi \rangle|^2}{\underbrace{\langle \psi | \psi \rangle}_{1}} = \left(\frac{-8\sqrt{15}}{(n\pi)^3} \right)^2 = \frac{64 \cdot 15}{(n\pi)^6}$$

$$\langle E \rangle_6 = \sum_{n=1}^{6} P_n E_n \text{ where } E_n = \frac{\hbar^2 \pi^2 n^2}{2\mu L^2}$$

~~Recap~~

$$\langle \phi_i | \psi \rangle = \frac{\sqrt{\frac{2}{L}}\,N}{n\pi}\left(2L^2\cos(n\pi) + L\cos(n\pi) - L\right)$$

$$\Rightarrow P_i = \frac{|\langle \phi_i | \psi \rangle|^2}{\underbrace{\langle \psi | \psi \rangle}_{1}} = \left[\frac{\sqrt{\frac{2}{L}}\,N}{n\pi}\left(2L^2\cos(n\pi) + L\cos(n\pi) - L\right)\right]^2$$

$$\langle E \rangle_6 = \sum_{n=1}^{6} P_n E_n \quad \text{where} \quad E_n = \frac{\hbar^2 \pi^2 n^2}{2\mu L^2}$$

$\Downarrow$

~~average energy using only the first 6 eigenstates of PIB~~

~~Since we were not given a definite value for L... this is the best we can do.~~

② $[\hat{H}, t] = \hat{H}\hat{t} - \hat{t}\hat{H}$

$\Downarrow$

act the commutator on an arbitrary function of time

$$(\hat{H}\hat{t} - \hat{t}\hat{H})f(t)$$

$$\Rightarrow i\hbar\left(\frac{\partial}{\partial t}t - t\frac{\partial}{\partial t}\right)f(t)$$

$$\Rightarrow i\hbar\left(\frac{\partial}{\partial t}(tf(t)) - t\frac{\partial}{\partial t}(f(t))\right)$$

$$\Rightarrow i\hbar\left(f(t) + \cancel{t\frac{\partial f}{\partial t}} - \cancel{t\frac{\partial f}{\partial t}}\right) = i\hbar f(t)$$

$$\Rightarrow [\hat{H}, t] = i\hbar$$

no they don't commute

③ Since

$$[\hat{H},\hat{t}] = i\hbar$$

and from Blinder we know

$$\Delta a \Delta b \geq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|$$

$$\Rightarrow \Delta E \Delta t \geq \frac{1}{2}\left|\langle[\hat{H},\hat{t}]\rangle\right|$$

$$\Rightarrow \Delta E \Delta t \geq \frac{1}{2}\left|i\hbar\right|$$

$$\Rightarrow \Delta E \Delta t \geq \frac{\hbar}{2}$$

$\Rightarrow$ In other words, a state ~~which~~ with a finite lifetime Δt has an associated uncertainty in its energy level given by $\Delta E \geq \frac{\hbar}{2\Delta t}$

④ Given $f(x)$, consider

$$e(x) = \frac{1}{\sqrt{2}}\left[f(x) + f(-x)\right] \text{ and } o(x) = \frac{1}{\sqrt{2}}\left[f(x) - f(x)\right]$$

Clearly

$$e(-x) = \frac{1}{\sqrt{2}}\left[f(-x) + f(x)\right] = e(x) \qquad o(-x) = \frac{1}{\sqrt{2}}\left[f(-x) - f(x)\right] = -o(x)$$

$\Downarrow$ even function $\qquad$ $\Downarrow$ odd function

orthogonal

$$\int_{-\infty}^{\infty} e(x)o(x)dx = \int_{-\infty}^{\infty} \tilde{o}(x)dx = 0$$

$\Downarrow$ odd × even = odd

$$\int_{-\infty}^{\infty} e(x)\,e(x)dt = \int_{-\infty}^{\infty} \frac{1}{2}\left(f^2(x) + 2f(x)f(-x) + f^2(-x)\right)dx = 1$$

Parity Relation

$$\hat{\imath}\, e(x) = e(-x) = 1\, e(x)$$

$$\hat{\imath}\, o(x) = o(-x) = -1\, o(x)$$